Electricity and Magnetism, Exam 1, 20/02/2020

10 questions

Write your name and student number on the answer sheet. Use of a calculator is allowed. You may make use of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, $\hat{\boldsymbol{x}}$ is the unit vector in the x-direction, and T is a scalar. In your handwritten answers, please indicate vectors (unit vectors) with an arrow (hat) above the symbol.

1. (5 points) Calculate the divergence of $\mathbf{v} = (r\cos\theta)\hat{\boldsymbol{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\boldsymbol{\phi}}$
   (problem 1.40) The divergence in spherical coordinates is

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = 5\cos\theta - \sin\phi.$$

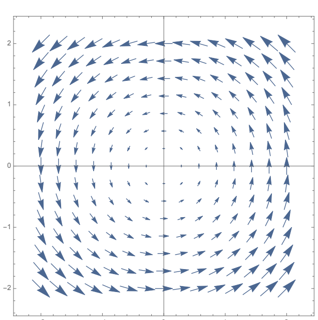
   +5 for correct answer, -2 when answer is a vector, -2 for major calculation errors, -1 for minor calculation errors

2. (5 points) Calculate the curl of $\mathbf{v} = x^2\hat{\boldsymbol{x}} + 3xz^2\hat{\boldsymbol{y}} + 2xz\hat{\boldsymbol{z}}$
   The curl is

$$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = -6xz\hat{\mathbf{x}} - 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}$$

   +5 for correct answer, -2 when forgetting vector identity, -2 for major calculation errors,-1 for missing a vector sign, -1 for minor calculation errors

3. (5 points) Give an example of a two-dimensional vector field with **zero divergence and non-zero curl**. Provide the formula of the vector field, explicitly calculate the divergence and curl, and make a sketch of the field.
   We need some rotation in the vector field, for example $\mathbf{F} = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$. *The curl* $\nabla \times \mathbf{F} = 2\hat{\mathbf{z}}$, *and the divergence* $\nabla \cdot \mathbf{F} = 0$.



+5 for correct answer,-1 for curving vectors, -1 for wrong curl, -1 for mistake in drawing

4. (5 points) Show how to calculate the volume of a cylinder of radius R and length l in cylindrical coordinates.

$$\begin{aligned} V = \int d\tau &= \int_{s=0}^{R} \int_{z=0}^{l} \int_{\phi=0}^{2\pi} s ds d\phi dz \\ &= \int_0^R s ds \int_0^{2\pi} d\phi \int_0^l dz \\ &= \frac{1}{2} R^2 2\pi l = \pi R^2 l \end{aligned}$$

+5 correct answer, +2 correct formula, +1 correct boundary, +2 correct execution

5. (5 points) $\int_0^2 (x^3 + 3x + 2)\delta(1 - x)dx =$
Only for $x = 1$ the delta function is non-zero; therefore, if $f(x) = x^3 + 3x + 2$ the answer is $f(1) = 6$
+5 correct answer, less points for making small mistakes

6. (5 points) Suppose $\mathbf{v} = \frac{1}{2r^2}\hat{\mathbf{r}}$. The divergence $\boldsymbol{\nabla} \cdot \mathbf{v} =$
The answer is $2\pi\delta^3(\mathbf{r})$
+5 correct answer, -1 for forgetting factor 2, 2 for 2π only answer, -1 for missing a vector sign on v, 0 points for $\nabla \cdot \mathbf{v} = 0$)

7. (5 points) Four charges are positioned on the corners of a square with size d. The two charges next to each other on one side of the square each have charge $+q$, the two others each have charge $-q$. What is the electric field (magnitude and direction) in the center of the square?
Assuming the side of the square has length d, we find the following. The distance of all charges to the center is $\sqrt{2}(d/2)$ (1pt). The electric field due to a single charge is therefore $\frac{1}{4\pi\epsilon_0}\frac{q}{d^2/2}$ (1pt). Due to the different angles, the charges all contribute a fraction of $1/\sqrt{2}$ in the same direction (1pt). In total we have (1pt)

$$\frac{4}{\sqrt{2}} \frac{1}{4\pi\epsilon_0} \frac{q}{d^2/2} = \frac{\sqrt{2}}{\pi\epsilon_0} \frac{q}{d^2},$$

pointing in the direction from side with the positive to side with the negative charges (1pt).

8. (5 points) Show how to derive the differential form of Gauss's law from the integral form by making use of the divergence theorem.
The divergence theorem states (1pt):

$$\int \mathbf{F} \cdot d\mathbf{a} = \int \nabla \cdot \mathbf{F} \, dV$$

Gauss' law in integral form is given by (1pt):

$$\int \mathbf{E} \cdot d\mathbf{a} = Q_{enc}/\varepsilon_0$$

Applying the divergence theorem to the left-hand-side we obtain (1pt):

$$\int \nabla \cdot \mathbf{E} \, dV = Q_{enc}/\varepsilon_0$$

However, the enclosed charge can be written as the volume integral over the charge density ρ as (1pt):

$$Q_{enc} = \int \rho \, dV$$

Plugging these results into the integral form of Gauss' law and equating the integrands gives (1pt):

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$$

9. (10 points) A thick spherical shell carries charge density $\rho = k/r^2$ with $a \leq r \leq b$. Find the electric field in the three regions (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$. Finally, plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.
Problem 2.15

Use a spherical Gaussian surface of radius r:
i) $\vec{E} = \vec{0}$ since $Q_{enc} = 0$
ii) $Q_{enc} = \int_a^r \int_0^{2\pi} \int_0^{\pi} \rho r^2 \sin\theta dr d\theta d\phi = 4\pi \int_a^r k dr = 4\pi k(r-a)$. Then, $\oint \vec{E} \cdot \vec{da} = 4\pi r^2 E_r = \frac{1}{\epsilon_0} 4\pi k(r-a)$ so $\vec{E} = \frac{k}{\epsilon_0} \frac{(r-a)}{r^2} \hat{r}$
iii) Q_{enc} becomes $4\pi k(b-a)$, hence: $\vec{E} = \frac{k}{\epsilon_0} \frac{(b-a)}{r^2} \hat{r}$

3 points for each subquestion and 1 point for graph. Deduct 1 or 2 points per subquestion for minor/major error respectively. Even if all calculations are wrong, 1 or 2 points can be earned for understanding of Gaussian surfaces.

10. (5 points) Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$. Argue what the field is in each of three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.
The field always points perpendicular to the planes. The field cancels in the outer regions and adds up in the middle. (+1 for correct direction, +1 per correct region (direction), +1 for giving the correct field strength in region (ii) ($\frac{\sigma}{\epsilon_0}\hat{n}$))

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